# Three-color Ramsey numbers for paths

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For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer $n$ such that if the edges of a complete graph $K_n$ are partitioned into $r$ disjoint color classes giving $r$ graphs $H_1, H_2, \ldots, H_r$, then at least one $H_i$ $(1 \leq i \leq r)$ has a subgraph isomorphic to $G_i$. The existence of such a positive integer is guaranteed by Ramsey's classical result. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ for $r \geq 3$ even for very special graphs. In this paper we consider the case when each $G_i$ is a path $P_n$ on $n$ vertices. For $r = 2$ a well-known theorem of Gerencsér and Gyárfás states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

For $r \geq 3$ the Ramsey numbers for $P_n$ are not known.

We prove - for sufficiently large n - the following conjecture of Faudree and Schelp.

$$R(P_n, P_n, P_n) = \begin{cases} 2n-1 & \text{for odd } n, \\ 2n-2 & \text{for even } n, \end{cases}$$

for the three-color Ramsey numbers of paths on $n$ vertices.

In an earlier version of this paper we proved the asymptotic result $R(P_n, P_n, P_n) = (2+o(1))n$. This was also obtained independently by Figaj and Łuczak. In the proof of the asymptotic result the main tools were the Regularity Lemma and the following lemma, suggested by Łuczak.

**Lemma 1.** *For every $\eta > 0$ there exists $n_0$ and $\varepsilon > 0$ such that for $n \geq n_0$ the following holds. For every 3-edge coloring of a graph $G$ with $n$ vertices and more than $(1-\varepsilon)\binom{n}{2}$ edges there exists a monochromatic connected matching covering at least $(1-\eta)n/2$ vertices of $G$.*

To obtain the exact result we sharpen Lemma **??**. This is a joint result of András Gyárfás, Gábor Sárközy, Endre Szemerédi and myself.